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1991 J. Phys. A: Math. Gen. 24 L165

(http://iopscience.iop.org/0305-4470/24/4/001)

J. Phys. A: Math. Gen. 24 (1991) L165–L168. Printed in the UK

LETTER TO THE EDITOR

A two-parametric quantization of sl(2)

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Received 12 October 1990

Abstract. A two-parametric solution of the constant Yang–Baxter equation is used for the construction of structures corresponding to quantized groups. The corresponding Hopf algebra is generated by five generators.

After the discovery of the quantized group $\mathrm{sl}_q(2)$ by Kulish and Reshetikhin (1981) and its reformulation in terms of a non-commutative Hopf algebra (Drinfeld 1986, Jimbo 1985) it was found by Reshetikhin *et al* (1989) that the defining relations for $\mathrm{sl}_q(2)$ can be obtained from the relations

$$R^+L_1(\varepsilon)L_2(\varepsilon)=L_2(\varepsilon)L_1(\varepsilon)R^+ \qquad R^+L_1(+)L_2(-)=L_2(-)L_1(+)R^+ \tag{1}$$

where

$$R^+:=PR_qP \qquad R_q:=\begin{pmatrix} q & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & \Omega & 1 & 0\\ 0 & 0 & 0 & q\end{pmatrix} \qquad P:=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{2}$$

$$\Omega=q-q^{-1} \qquad \varepsilon=\pm \qquad L_1(\varepsilon)=L(\varepsilon)\otimes 1 \qquad L_2(\varepsilon)=1\otimes L(\varepsilon) \tag{3}$$

and

$$L(+)=\begin{pmatrix} K & E\\ 0 & M\end{pmatrix} \qquad L(-)=\begin{pmatrix} M & 0\\ F & K\end{pmatrix}. \tag{4}$$

Indeed, inserting (2)–(4) into (1) we obtain the relation for the 'quantized universal enveloping algebra' $U_q(\mathrm{sl}(2))$

$$\begin{gathered} KM=MK \qquad KF=qFK \qquad KE=q^{-1}EK\\ EF-FE=(K^2-M^2)\Omega \qquad MF=q^{-1}FM \qquad ME=qEM. \end{gathered} \tag{5}$$

The algebra $U_q(\mathrm{sl}(2))$ is obtained as the free algebra generated by (K, M, E, F) and factorized by the ideal corresponding to (5).

The familiar commutation relations of $\mathrm{sl}_q(2)$

$$[J_0, J_\pm]=\pm J_\pm \qquad [J_+, J_-]=\frac{\sinh(2\eta J_0)}{\sinh\eta} \tag{6}$$

can be obtained from (5) by substitution

$$q = e^{\eta} \qquad K = q^{-J_0} \qquad M = q^{J_0} \qquad E = \Omega J_+ \qquad F = -\Omega J_-. \tag{7}$$

This substitution also implies

$$KM = MK = 1 \tag{8}$$

and it can be shown that the algebra $U_q(\mathrm{sl}(2))$ defined by the relations (5) and (8) is a (non-commutative) Hopf algebra.

Besides that, it is possible to define an algebra of functions on the 'quantum matrix algebra' $\mathrm{Fun}(\mathrm{sl}_q(2))$ by inserting $R = R_q$ into

$$RT_1T_2 = T_2T_1R \qquad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{9}$$

and 'quantum vector spaces' by

$$f(PR)(x \otimes x) = 0 \tag{10}$$

(see Reshetikhin *et al* 1989). The matrix R_q is a solution of the constant Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{11}$$

In fact R_q is a special form of a two-parametric solution found in Hlavatý (1987) that can be written as

$$R_{q,s} := \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & \Omega & s^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{12}$$

Obviously $R_{q,s=1} = R_q$ and a natural question is: what quantized algebras are obtained when $R_{q,s}$ is used in the above-sketched derivation? The answer is the content of this letter.

Inserting $R^+ = PR_{q,s}P$ into the relation (1) we obtain a modification of the defining relations for the quantized universal enveloping algebra in the form

$$\begin{aligned} &KM = MK \qquad KF = qsFK \qquad KE = (qs)^{-1}EK \\ &s^{-1}EF - sFE = (K^2 - M^2)\Omega \qquad MF = q^{-1}sFM \qquad ME = qs^{-1}EM. \end{aligned} \tag{13}$$

We shall call the algebra defined by these relations $U_{q,s}(\mathrm{sl}(2))$. By the substitution

$$K = (qs)^{-J_0} \qquad M = (q^{-1}s)^{-J_0} \qquad E = \Omega J_+ \qquad F = -\Omega J_- \tag{14}$$

we can get a modification of the commutation relation (6) in the form

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-]_s := s^{-1}J_+J_- - sJ_-J_+ = s^{-2J_0}\frac{\sinh(2\eta J_0)}{\sinh \eta}. \tag{15}$$

Several comments on the properties of this two-parametric deformation of sl(2) are in order.

(i) Similarly to $\mathrm{sl}_q(2)$, the Pauli matrices are the two-dimensional representation of the algebra (15).

(ii) The comultiplication Δ defined by

$$\begin{aligned} &\Delta(K)=K\otimes K \qquad \Delta(M)=M\otimes M \\ &\Delta(E)=K\otimes E+E\otimes M \qquad \Delta(F)=K\otimes F+F\otimes M \end{aligned} \tag{16}$$

preserves relations (13) and converts the algebra $U_{q,s}(\mathrm{sl}(2))$ into a bialgebra.

(iii) Differently from the $\mathrm{sl}_q(2)$, the element KM does not belong to the centre of $U_{q,s}(\mathrm{sl}(2))$ and therefore cannot be identified with the unit. This is a main obstacle that prevents finding an antipode function on the bialgebra $U_{q,s}(\mathrm{sl}(2))$ and converting $U_{q,s}(\mathrm{sl}(2))$ into a Hopf algebra.

For this reason the algebra determined by (13) should rather be called a 'quantum semigroup'.

We can also define the algebra $\mathrm{Fun}(\mathrm{sl}_{q,s}(2))$ of functions on the quantum matrix algebra and the quantum vector spaces $C^2_{f(R_{q,s})}$. The algebra $\mathrm{Fun}(\mathrm{sl}_{q,s}(2))$ is determined by the relation (9) where $R=R_{q,s}$. This yields

$$\begin{aligned} &AB=BAqs \qquad CB=BCs^2 \qquad AC=CAqs^{-1} \\ &BD=DBqs^{-1} \qquad CD=DCqs \qquad [A,D]=(qs^{-1}-q^{-1}s^{-1})CB. \end{aligned} \tag{17}$$

It is easy to define the bialgebra structure on $\mathrm{Fun}(\mathrm{sl}_{q,s}(2))$ by

$$\Delta(T_i^j)=T_i^k\otimes T_k^j \tag{18}$$

but determination of the Hopf structure on this algebra is not as simple as for the algebra associated with R_q. Nevertheless, according to the theorem 8 in Reshetikhin *et al* (1989) it should be possible to define the Hopf structure because the matrix $R_{q,s}$ fulfils also the condition

$$\Gamma(R_{q,s})P\Lambda(R_{q,s})PK_0=\text{constant } K_0 \tag{19}$$

where

$$\Gamma(R_{q,s}):=(R_{q,s}^{-1})^{t_1} \tag{20}$$

$$\Lambda(R_{q,s}):=(R_{q,s}^{t_2})^{-1} \tag{21}$$

(t_1, t_2 are transpositions in the first and second indices of the matrix $R=(R^{j_1j_2}_{i_1i_2})$) and

$$K_0:=\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}.$$

For definition of the quantum vector spaces it is important that the matrix $\tilde{R}_{q,s}=PR_{q,s}$ satisfies the Hecke condition

$$(\tilde{R}_{q,s})^2:=\Omega(\tilde{R}_{q,s})+1. \tag{22}$$

This makes it possible to write the matrix $R_{q,s}$ in terms of two projectors

$$P^+:=\frac{(\tilde{R}_{q,s})+q^{-1}}{q+q^{-1}} \qquad P^-:=\frac{(-\tilde{R}_{q,s})+q^{-1}}{q+q^{-1}} \tag{23}$$

of ranks 3 and 1. These projectors can be used to define 'the algebra of functions on the two-dimensional quantum vector space' (Reshetikhin *et al* 1989) by generators x_1, x_2 satisfying

$$x_1 x_2 = q x_2 x_1 \tag{24}$$

and 'the exterior algebra of the two-dimensional quantum vector space' defined by

$$(y_1)^2 = (y_2)^2 = 0 \qquad s y_1 y_2 = -q^{-1} y_2 y_1. \tag{25}$$

As noted before, it seems that $U_{q,s}(\mathrm{sl}(2))$ cannot be turned into a Hopf algebra. However, this can be done if we extend the algebra by another generator Z and the following relations

$$KZ = ZK \qquad MZ = ZM \qquad ZE = s^2 EZ \qquad ZF = s^{-2} FZ. \tag{26}$$

The element KMZ then belongs to the centre and we can set

$$KMZ = 1. \tag{27}$$

The resulting algebra is a bialgebra with comultiplication given by (16) and

$$\Delta(Z) = Z \otimes Z. \tag{28}$$

It can be turned into the Hopf algebra by definitions of the antipod and counit functions as

$$\begin{aligned} &S(K) = MZ \qquad S(M) = KZ \qquad S(Z) = KM \qquad S(E) = -qsEZ \\ &S(F) = -(qs)^{-1} FZ \qquad \varepsilon(K) = \varepsilon(M) = \varepsilon(Z) = 1 \qquad \varepsilon(E) = \varepsilon(F) = 0. \end{aligned} \tag{29}$$

This extension of $U_{q,s}(\mathrm{sl}(2))$ into the Hopf algebra then can be considered as the dual object (Drienfeld 1986) to a quantum group that is a two-parametric deformation of the group sl(2).

Let us note finally that the matrix (12) is not the only solution of the constant YBE found in Hlavatý (1987). Besides this, there are nine others. Unfortunately, most of them do not converge to the unit matrix for special values of parameters. It means that the algebras they define have no 'classical counterparts'. Nevertheless, it might be interesting to investigate their properties.

Useful discussions with M Bednář and M Havlíček are gratefully acknowledged.

Note added in proof. After finishing this letter we became aware of the preprint of Schirrmacher *et al* (1990) where the matrix (12) is used but the quantization procedure is different. Moreover, from the referee's report we found out that the algebra that is investigated in detail here is a special case of two-parameter quantization of gl(n) (Takeuchi 1990), which is a special case of multiparameter quantization of gl(n) (Sudbery 1990), which is a special case of multiparameter quantization of semisimple Lie algebras (Reshetikhin 1990).

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